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PERTURBED BUTTERWORTH POLE PATTERNS FOR TRACKING IN THE SENSE OF SPHERES

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The so called "Quantitative Pole Placement"(QPP) identified in the context of guaranteed tracking in the sens of spheres is considered. In the prior literature this pole-placement problem was treated in a somewhat adhoc way. The primary purpose of the present work therefore is to propose a systematic procedure for such pole placement. The approach to the problem is based on a generalization of the standard LQ problem formulation. The preferred pole locations that minimize a crucial operator norm needed for the success of the QPP formulation are shown to be a perturbed version of the Butterworth pole configuration. The results are applied to a 3 d.o.f. robotic manipulator for illustrating the evolving methodology. At the center of the overall design philosophy is the need to directly satisfy performance specifications in uncertain, nonlinear systems.

Key Words: Quantitative Pole Placement, Tracking in the Sese of Spheres, Generalized LQ Formulation, L_∞-norm, Banach Contraction Mapping, Butterworth Pole Configuration.

1. INTRODUCTION

The purpose of this paper is to illustrate an approch for the systematic placemet of eigenvalues for the so called "Quantitative Pole Placement"(QPP) problem formulated in the context of "tracking in the sense of spheres", first introduced in Barnard and Jayasuriya (Barnard et al., 1982; Jayasuriya et al., 1988; Jayasuriya et al., 1984; Kee, 1987). These formulations have been motivated by the need for a formal mathematical synthesis procedure for the direct satisfication of design specifications in the presence of uncertain plant dynamics. The work of Horowitz (Horowitz, 1963; Horowitz, 1967; Horowitz, 1976; Horowitz, 1982) is unique with respect to this design philosophy. Somewhat related work with respect to stability and tracking include Leitmann (Leitmann, 1979; Leitmann, 1982) and his co-workers, Usoro, et. al. (Usoro et al., 1982) and the more recent work by Zames (Zames, 1981) and ohers.

The main desig criterion central to the methodology of the controller for tracking in the sense of spheres (Barnard, 1980; Barnard et al., 1982; Jayasuriya et al., 1984) can be stated as a QPP procedure for adjusting the size of a certain linear opeator norm. In order to view this QPP in the correct perspective we first describe the conventional pole-placement problem by considering a linear time invariant system of the form

$$\dot{x}(t)=Ax(t)+Bu(t)$$

where, $x(t)\in R^n, u(t)\in R^m$, and A and B are constant matrices of appropriate dimensions. If (A,B) is controllable then arbitrary eigenvalues for the closed loop system can be achieved by applying the state feedback $u(t)=Kx(t)$ to this system. This idea of arbitrary placement of closed loop eigenvalues is what is typically referred to as pole-placement. Typically this is achieved by a trial and error procedure.

In the linear-quadratic (LQ) problem formulation once the weighting matrices of the performance index (PI) are chosen, the eigenvalue locations for the closed loop system can be computed by solving the Riccati equation. Although the eigenvalue locations depend on the weighting matrices the LQ formulation affords a way of selecting the closed loop eigenvalues formally. This formulation is the motivation for the approach described in this paper for the QPP, and is based on a generalized LQ formulation. The latter is achieved by considering a linear system characterizing the closed loop linear operator pivotal to the satisfaction of the design criteria. (Jayasuriya et al., 1984). As a consequence of the generalized LQ formulation Butterworth type pole configurations appear to yield satisfactory QPP. Therefore the main contribution of this paper is, in executing a particular controller design the algorithms for its synthesis are initiated by selecting the closed loop poles in a Butterworth pattern.

The paper is organized as follows. In section Ⅱ the design criteria for the tracking in the sense of spheres are introduced to highlight the need for QPP. The generalization of the LQ problem is considered in section Ⅲ. Then in section Ⅳ we apply the results to a 3 d.o.f robotic manipulator followed by coclusions in section Ⅴ.

2. QUANTITATIVE POLE PLACEMENT

Design criteria for the tracking in the sense of spheres (Jayasuriya et al., 1984) can be extended to systems modeled

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by state and output equations of the form

$$\dot{x}(t)=Ax(t)+B(t)+Dw(t)+f(x(t),\gamma,t) \quad (1a)$$
$$y(t)=Cx(t) \quad (1b)$$

where the state $x(t)\in R^n$, the control $u(t)\in R^m$, the uncertainty $\gamma\in\Gamma\subset R^a$, the time $t\in T=[0,\infty]$, the external disturbance $w(t)\in W\subset R^d$, the nonlinear function $f: R^n\times R^a\times T\rightarrow R^n$, and the output$(t)\in R^b$, the pair$(A,B)$ is completely controllabel, and the pair$(A,C)$is completely observerble. The matrices A,B,C and D are constant matrices of dimension $n\times n, n\times m, b\times n$,and $n\times d$ respectively.

A Luenberger type nonlinear state observer of the form

$$\begin{aligned}\dot{\hat{x}}(t)&=A\hat{x}(t)+Bu(t)+f_0(\hat{x}(t),t)+GC(\hat{x}(t)\\&\quad-x(t))+V_1r(t)\\u(t)&=V_2r(t)+K\hat{x}(t)\end{aligned} \quad (2)$$

where $\hat{x}(t)\in R^n$, $u(t)\in R^m, t\in T=[0,\infty)$, reference input $r(t)\in R^m$, $f_0: R^n\times T\rightarrow R^n$ and G,K,V_1 and V_2 are constant matrices of appropriate dimensions, is employed to implement nonlinear state feedback control.

The feedback system represented by Eq. (1) and Eq. (2) are combined into the form

$$\begin{aligned}\dot{z}(t)&=Rz(t)+B_0r(t)+B_1w(t)+\xi(z,(t),\gamma,t)\\y(t)&=C_0z(t)\end{aligned} \quad (3)$$

where

$$z(t)=\begin{bmatrix}x(t)\\\hat{x}(t)\end{bmatrix}\in R^{2n}$$
$$R=\begin{bmatrix}A & BK\\-GC & A+BK+GC\end{bmatrix}$$
$$B_0=\begin{bmatrix}BV_2\\BV_2+V_1\end{bmatrix}$$
$$B_1=\begin{bmatrix}D\\0\end{bmatrix}$$
$$C_0=[C \quad 0]$$
$$\xi(z(t),\gamma,t)=\begin{bmatrix}f(x(t),\gamma,t)\\f_0(x(t),\ t)\end{bmatrix}$$

Assuming that the function $\xi(z(t),\gamma,t)$ is continuons, and the eigenvalues of the matrix R are in the open left half complex plane, we can write the Eq. (3) in an equivalent operator form using Laplace transform

$$\begin{aligned}\dot{z}(t)&=\Psi N_\gamma z(t)+\Psi B_0r(t)+\Psi_{B1}(t)+q(t)\\y(t)&=C_0z(t)\end{aligned} \quad (4)$$

where

Ψ and N_γ are respectively a linear and an uncertain nonlinear map from $L_\infty^{2n}(T)$ back into itself given by

$$(\Psi z)(t)=\int_0^t e^{R(t-\tau)}z(\tau)d\tau$$
$$(N_\gamma z)(t)=\xi(z(t),\gamma,t)$$
$$q(t)=e^{Rt}z(0),$$

Next, by considering a corresponding dynamical system which is completely known (i.e. free of uncertain elements). the nominal operator equations can be written as

$$\dot{z}_0=\Psi N_0z_0(t)+\Psi B_0r_0(t)+q(t) \quad (5)$$
$$y_0=C_0(z_0)$$

where (r_0,y_0,z_0) is a completely known triple of a specified nominal output y_0, and corresponding solutions z_0,r_0, with r_0 serving as a nominal command input relative to y_0. The nonlinear map N_0 is represented by

$$(N_0z)(t)=\begin{bmatrix}f_0(x(t),t)\\f_0(\hat{x}(t),t)\end{bmatrix}$$

To compare the actual and nominal systems for any uncertain combination (w,r,z,y) satisfying Eq. (4) and a known combination (r_0,y_0,z_0) satisfying Eq. (5), the differences

$$z-z_0=\Psi(N_\gamma z-N_0z_0)+\Psi B_0(r-r_0)+\Psi B_1w$$
$$y-y_0=C_0(z-z_0)$$

are transformed into fixed point formulation

$$\begin{aligned}\bar{z}&=\phi\bar{z}\\&=W_0\Psi(N_\gamma W_0^{-1}\bar{z}-N_0W_0^{-1}\bar{z}_0)+W_0\Psi B_0(r-r_0)\\&\quad+W_0\Psi B_1w+\bar{z}_0\\y-y_0&=C_0W_0^{-1}(\bar{z}-\bar{z}_0)\end{aligned} \quad (6)$$

where $\phi: L_\infty^{2n}(T)\rightarrow L_\infty^{2n}(T)$ is a nonlinear map, W_0 is an arbitrary nonsingular weighting matrices, and

$$\bar{z}=W_0z,$$
$$\bar{z}_0=W_0z_0$$

By applying the Banach contraction mapping theorem to Eq. (6), we obtain the following result which gives sufficient condition on design elements G,K,V_1,V_2, and the design function f_0 that assure servo-tracking in the sense of input-output spheres.

Remarks:

(1) The norm considered throughout the paper is the L_∞-norm unless stated otherwise and is denoted by $\|\cdot\|$.

(2) A given output(input) $\{g: T\rightarrow R^b\}\in L_\infty^b(0,\infty)$, is said to belong to an output(input) sphere $\Omega(g: g_0,\beta)$ of radius $\beta>0$ centered at $\{g_0: T\rightarrow R^b\}\in L_\infty^b(0,\infty)$ if $\|g-g_0\|\leq\beta$. g_0 is referred to as the nominal output(input) and $\Omega(g: g_0,\beta)=\{g\mid\|g-g_0\|\leq\beta\}$. Here $T=(0,\infty)$ is the time set.

(3) If the system output y lies in the output sphere $\Omega(y: y_0,\beta_0)$ for any input sphere $\Omega(r: r_0,\beta_1)$ then the system is said to track y_0 "in the sense of input-output spheres"

Theorem 1: Let f and f_0 be continuous, and let G and K be assigned so that the eigenvalues of matrix R are in the open left-hand complex plane. Let (r_0,y_0,z_0) be a known combination satisfying Eqs. (5) and (r,y,z,w) be any combination satisfying Eqs.(4). Then for any input r in the specified sphere

$$\Omega(r: r_0,\beta_i)=\{r\in L_\infty^m\mid\|r-r_0\|\leq\beta_i\}$$

and for any external disturbances w in the specified sphere

$$\Omega(w: 0,\beta_w)=\{z\in L_\infty^d\mid\|w\|\leq\beta_w\}$$

there exists a unique combined response z in the specified β_0-neighbourhood

$$\Omega(z: z_0,\beta_0)=\{z\in L_\infty^{2n}\mid\|W_0(z-z_0)\|\leq\beta_0\}$$

provided $\eta \leq \dfrac{\beta_0}{\rho_0\beta_i+\rho_1\beta_w+\rho_2+\rho_3\beta_0}$ (7)

where

$$\eta = \| W_0 \Psi Q_0^{-1} \|$$
$$\rho_0 = \| Q_0 B_0 \|$$
$$\rho_1 = \| Q_0 B_1 \|$$
$$\rho_2 = \sup_{\gamma \in \Gamma} \| Q_0 [N_\gamma z_0 - N_0 z_0] \|$$
$$\rho_3 = \sup_{\substack{z,z' \in \Omega(z\,:\,z_0,\beta_0) \\ z \neq z \\ \gamma \in \Gamma}} \frac{\| Q_0 [N_\gamma z - N_\gamma z'] \|}{\| W_0 (z-z') \|}$$

with respect to a nonsingular constant weighting matrix Q_0.

Remarks: Inequality (7) will be referred to as the primary design criterion for precision tracking in the sense of spheres. Some important design features of this criterion are

(i) Design elements G,K,V_1,and V_2 must be chosen so that the eigenvalues of the matrix $R \in R^{2n\times 2n}$ are at suitable locations in the open left-half complex plane and that the inequality (7) of the Theorem 1 is satisfied. The upper bound on the operator norm $\| W_0 \Psi Q_0^{-1} \|$ depends on the design specifications such as tracking accuracy, the extent of the disturbances and the size of the uncertainties. Thus for precise traking in the presence of large disturbances and large plant uncertainties, a small value of operator norm $\| W_0 \Psi Q_0^{-1} \|$ is typically needed which might result in high gain feedback. It should be noted however, that the norm bound requirement is only a sufficient condition.

(ii) A larger upper bound for the linear operator norm $\| W_0 \Psi Q_0^{-1} \|$ can be allowed by a proper choice of a nonlinear design function f_0. The norm values ρ_2,ρ_3 and η play a vital role in the design procedure. ρ_2 can be regarded as a measure of the maximal difference of operator N_γ and operator N_0 at z_0, ρ_3 is a measure of the severity of the nonlinearity and the uncertainty of the system, and η is a measure of the "trackability" of the system. The function f_0 should be assigned so that the values of ρ_2 and ρ_3 will allow a larger upper bound for the linear operator norm $\| W_0 \psi Q_0^{-1} \|$.

(iii) A Quantitiative pole-placement (QPP) is defined by (7). That is, a proper selection of eigenvalues of the matrix R will potentially enable one to satisfy the operator norm condition. To achieve an optimum design the eigenvalues must be placed so that the operator norm is as close as possible to the threshold value,

$$\frac{\beta_0}{\rho_0\beta_i+\rho_1\beta_w+\rho_2+\rho_3\beta_0}$$

(iv) The nominal plant defined in Eq. (5) is an essential part of the design criteria. That is, the nominal input r_0 must be determined so that (r_0,y_0) satisfies the nominal equations.

3. GENERALIZED LQ PROBLEM

We first recall the LQ results for the regulator problem to motivate the generalized LQ problem for the QPP described in the previous section.

For a plant given by the state equations

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (8a)$$
$$y(t) = Cx(t) \quad (8b)$$

where (A,B) is controllable, the optimal controller which minimizes the quadratic performance index (PI)

$$J = \int_0^\infty [x^T S x + u^T W u] dt \quad (9)$$

where S and W are symmetric positive weighting matrices and u is given by

$$u = -Kx \quad (10)$$

where the controller gains are computed as

$$K = W^{-1} B^T K_1 \quad (11)$$

with $K_1 = K_1^T > 0$ as the unique solution of the algebraic Riccati equation

$$0 = -K_1 A - A^T K_1 - S + KBW^{-1} B^T K_1 \quad (12)$$

It is important ot note here that once S and W in the PI (9) are specified, K_1 can be determined by Eq. (12). Then K_1 in turn determines the state feedback gains of the control law (10) which corresponds to a definite closed loop pole-configuration for the system given by (8). The ability to generate this somewhat definitive pole-configuration is the spirit in which we develop the generalized LQ problem for the QPP.

First we observe that the operator norm $\eta = \| W_0 \Psi Q_0^{-1} \|$ crucial to the QPP depends only on the linear structure of the uncertain nonlinear plant (1). The linear part of the plant is given by Eq. (8).

Recalling the fact that for $f(t) \in L_p[1,\infty)$
$\|f\|_p = [\int_0^\infty |f(t)|^p dt]^{1/p}$ for $p \in [1,\infty)$

and that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty = \sup_{t\in[0,\infty)} |f(t)|,$$

we modify the LQ performance measure (8) as

$$J_p = \int_0^\infty \Big[\frac{1}{2p} \{(y(t)-y_0(t))^T Q_1 (y(t)-y_0(t))\}^p \\ \frac{1}{2p} \{u(t)^T Q_2 u(t)\}^p \Big] dt \quad (13)$$

to reflect an analogous L_∞-measure in the limit as $p\to\infty$. In the PI given by (13), $y_0(t) \in R^b$ is the nominal output and Q_1, Q_2 are repectively, symmetric positive definite weighting matrices of order $b\times b$ and $m\times m$.

Next, by standard variational arguments we obtain the following conditions for the optimal contorl $\tilde{u}(t)$ that steers the system output in such a way as to track the nominal output $y_0(t)$ simultaneously minimizing the PI (13) (Kwakernaak et al., 1972).

(i) $\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t)$ (state equation) (14)

(ii) $\dot{\tilde{v}} = \{(C\tilde{x}-y_0)^T Q_1 (C\tilde{x}-y_0)\}^{p-1} C^T Q_1 (C\tilde{x}-y_0) - A^T v$ (costate equation) (15)

(iii) $\{\tilde{u}^T Q_2 \tilde{u}\}^{p-1} Q_2 \tilde{u} + B^T \tilde{v} = 0$ (16)

(iv) $\tilde{v}(T_f) = 0$ (17a)

$H(\tilde{x}(T_f), \dot{\tilde{x}}(T_f), \tilde{u}(T_f), \tilde{v}(T_f), T_f) = 0$ (17b) where the Hamiltonian

$$H=\frac{1}{2p}\{(Cx-y_0)^TQ_1(Cx-y_0)\}^p+\frac{1}{2p}\{u^TQ_2u\}^p+v^T(Ax+Bu).$$

andrepresents the optimality and T_f is the final time.

Equations(14) ~ (17) constitute a set of necessary conditions for an extremal of the generalized LQ performance index. If $p=1$ we recover the LQ results in the form of a state feedback with gains given by the Riccatti differential equation. In order to obtain a perturbed form of this LQ solution or equivalently the LQ pole-patterns for the generalized LQ problem (13) we start by defining the positive quantities

$$\lim_{p\to\infty}\{(C\tilde{x}-y_0)^TQ_1(C\tilde{x}-y_0)\}^{p-1}=\epsilon_1(t) \tag{18}$$

$$\lim_{p\to\infty}\{\tilde{u}^TQ_2\tilde{u}\}^{p-1}=\epsilon_2(t) \tag{19}$$

which are then substituted in (15) and (16). The costate Eq. (15) then become

$$\dot{\tilde{v}}=-\epsilon_1(t)C^TQ_1(C\tilde{x}-y_o)-A^T\tilde{v} \tag{20}$$

and the control $\tilde{u}(t)$ from(16) is

$$\tilde{u}(t)=-\frac{1}{\epsilon_2(t)}Q_2^{-1}B^T\tilde{v} \tag{21}$$

Substituting (21) into (14) and augmenting it with (20) yield,

$$\dot{\tilde{x}}(t)=A\tilde{x}-\frac{1}{\epsilon_2(t)}BQ_2^{-1}B^Tv \tag{22a}$$

$$\dot{\tilde{v}}(t)=-\epsilon_1(t)C^TQ_1C\tilde{x}-A^T\tilde{v}+\epsilon_1(t)C^TQ_1y_0 \tag{22b}$$

Rewriting Eq. (22) in a matrix form, given

$$\dot{\tilde{z}}=\tilde{A}\tilde{z}+\tilde{B}\tilde{u}_c \tag{23}$$

where

$$\tilde{z}=\begin{bmatrix}\tilde{x}\\ \tilde{v}\end{bmatrix}\in R^{2n}$$

$$\tilde{A}=\begin{bmatrix}A & -\frac{1}{\epsilon_2}BQ_2^{-1}B^T\\ -\epsilon_1C^TQ_1C & -A^T\end{bmatrix}\in R^{2n\times 2n}$$

$$\tilde{B}=\begin{bmatrix}0\\ I\end{bmatrix}\in R^{2n}$$

$$\tilde{u}_c=\epsilon_1C^TQ_1y_0\in R^m$$

Remarks : Equations(22) can be rewritten as

$$\dot{x}=A\tilde{x}-\frac{1}{\epsilon_2}BQ_2^{-1}B^T\tilde{v} \tag{24a}$$

$$\frac{1}{\epsilon_1}\dot{\tilde{v}}=-C^TQ_1C\tilde{x}-\frac{1}{\epsilon_1}A^T\tilde{v}+C^TQ_1y_0 \tag{24b}$$

and we observe that for ϵ_1 very large (or $\epsilon_1\to\infty$) by setting $\frac{1}{\epsilon_1}=0$ formally in (24b) that

$$C^TQ_1(C\tilde{x}-y_o)=0$$

This implies that $C\tilde{x}\to y_0$ which in a sense captures the continuous tracking requirement. This somewhat heuristic argument is the motivation for the limiting computations given below. To facilitate these computations we arbitrarily set $\frac{1}{\epsilon_2}Q_2^{-1}=1$ for the SISO case.

Now we derive the characteristic polynomial of $\tilde{A}$ as a function of ϵ_1 and ϵ_2, observing that the optimal closed loop poles which are the eigenvalues of the matrix $\tilde{A}$ should lie in the open left half complex plane. Moreover we set $\frac{1}{\epsilon_2}Q_2^{-1}=1$ for the SISO case as per the remarks made earlier.

It can be easily shown that

$$\det(sI-\tilde{A})=\det\begin{bmatrix}sI-A & BB^T\\ \epsilon_1C^TQ_1C & sI+A^T\end{bmatrix} \tag{25}$$

$$=(-1)^na(s)a(-s)\det\,(I+\epsilon_1h(-s)^TQ_1h(s)) \tag{26}$$

where

$$a(s)=\det(sI-A)$$
$$h(s)=C(sI-A)^{-1}B$$

and I's are the identity matrices of appropriate dimensions. Thus the eigenvalues of the closed loop system are the zeros of (26) which are in the open left half complex plane. Now we let the open loop transfer function $h(s)$ of a SISO system be represented by

$$h(s)=\frac{b(s)}{a(s)}=\frac{b_0\prod_{i=1}^{r}(s-\varphi_i)}{\prod_{i=1}^{n}(s-\lambda_i)} \tag{27}$$

where b_0 is a nonzero constant, $\varphi_i, i=1\cdots\cdots r$, are the zeros of the open loop system and λ_i, $i=1\cdots\cdots n$, are the poles of the open loop system, then with $Q_1=1$ for simplicity, (26) becomes

$$\prod_{i=1}^{n}(s-\lambda_i)(s+\lambda_i)+(-1)^{n-r}\epsilon_1b_0^2\prod_{i=1}^{r}(s-\varphi_i)(s+\varphi_i)=0 \tag{28}$$

The asymptotic behavior of these closed loop poles as $\epsilon_1\to\infty$ is given in the following theorem(Kwarkernaak et al., 1972).

Theorem 2 : Suppose that the open loop system is represented by the transfer function (27), then for $\epsilon_1\to\infty$, r eigenvalues of the closed loop system approach asymtotically the values $\tilde{\varphi}_i, i=1,\cdots r$,where

$$\tilde{\varphi}_i=\begin{cases}\varphi_i & \text{if } Re(\varphi_i)\le 0\\ -\varphi_i & \text{if } Re(\varphi_i)>0\end{cases}$$

and the remaining $(n-r)$ eigenvalues approach the asymptotes through the origin and make angles θ with the negative real axis of

(a) $\theta=\pm\frac{\ell\pi}{n-r}$, $\ell=0\cdots,\frac{n-r-1}{2}$, for $(n-r)$ odd

(b) $\theta=\pm\dfrac{(\ell+\frac{1}{2})\pi}{n-r}$, $\ell=0\cdots,\dfrac{n-r-2}{2}$, for $(n-r)$even.

The distance from the origin for the far away eigenvalues are asymptotically $(\epsilon_1 b_0^2)^{1/(2(n-r))}$.

Figures 1(a) and 1(b) show two Butterworth pole configurations corresponding to $(n-r)=2$, and 3 respectively.

In order to illustrate that the Butterworth pole configuration given in the above theorem leads to an almost minimum norm for the linear operator characterizing the closed loop system in QPP we give two examples.

Example: Consider the second order system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0.23 & 0.\\ -0.57 & 1.42\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u(t) \tag{29}$$

The set of eigenvalues for the closed loop system are

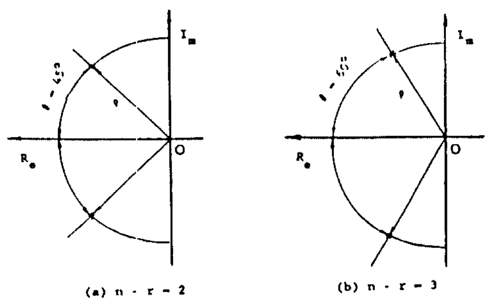

Fig. 1 Optimal pole configuration

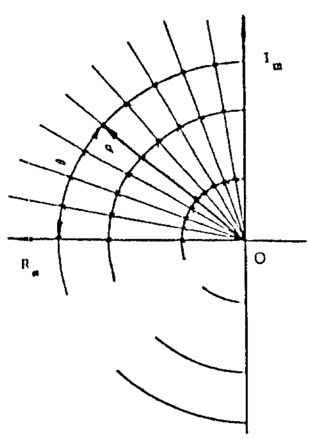

Fig. 2(a) Eigenvalue placement for n-2

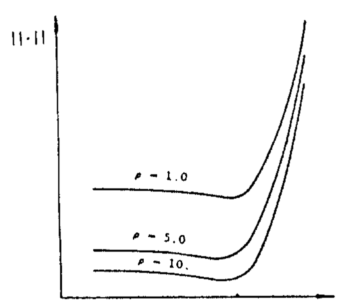

Fig. 2(b) Norm configuration for n-2

chosen so that the distance from the origin ρ, and the angle θ measured from the negative real axis shown in Fig. 2(a) are

ρ:	0.5	1.0	1.5	2.0	3.0	5.0	10.0	20.0
θ:	10.	20.	30.	40.	50.	50.	70.	80.

we then compute the closed loop operator norm $\|\Psi\|$ corresponding to each pole configuration specified by ρ and θ. The results shown in Fig. 2(b) verify that the minimum norm is obatined when the set of eigenvalues for the closed loop system are chosen in the vicinity of a 45° line. This is quite consistent with the predictions of the theorem given previously.

In the following section we apply these results from Theorems 1 and 2 to a robotic manipulator problem merely for the purpose of illustrating our design methodology.

4. 3 DOF MANIPULATOR

We consider the three DOF manipulator shown in Fig. 3. This manipulator has a rotational joint and a translational joint in the (x,y) plane. Moreover the arm can be lifted along the vertical z-axis thus defining the third degree of freedom.

The dynamic equations for this robot configuration follow directly from an application of Lagrange's equations and take the following form (Freund 1982)

$$M(\Psi(t),\gamma)\ddot{\Psi}(t)=-f(\Psi(t),\dot{\Psi}(t),\gamma)+u(t) \tag{30}$$

where $\Psi(t)=[r(t),\theta(t),z(t)]^T$ specifies the configuration at time t in a cylindrical frame of reference, γ is the payload uncertainty and the dot denotes time derivatives. $u(t)$ represents the generalized forces and is given by

$$u(t)=[F_r,T_\theta,F_z]^T$$

where F_r is the radial force, T_θ is the torque and F_z is the vertical force associated with the coordinates r,θ,and z respectively. $M(\Psi(t),\gamma)$ and $f(\Psi(t),\dot{\Psi}(t),\gamma)$ are given belows.

$$M(\Psi(t),\gamma)=\begin{bmatrix}(M_1+M_2) & 0\\ 0 & (M_1+M_2)r^2(t)-M_1\ell r(t)+J\\ 0 & 0\end{bmatrix}$$

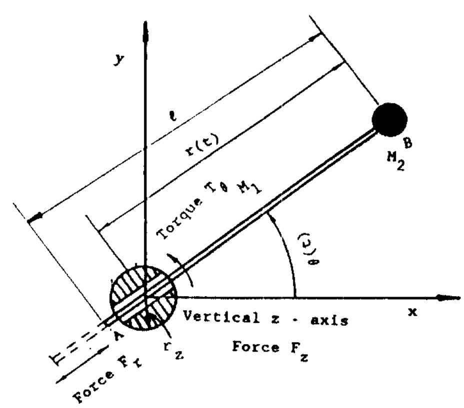

Fig. 3 A three DOF robot manipulator

$$\begin{matrix} 0 \\ 0 \\ (M_1+M_2) \end{matrix} \Bigg] \tag{31a}$$

$$f(\varphi(t),\dot{\varphi}(t),\gamma)=\begin{bmatrix} -M_{12}r(t)\dot{\theta}^2(t)+\frac{1}{2}M_1\ell\theta^2(t) \\ \{2M_{12}r(t)\dot{r}(t)-M_1\ell\dot{r}(t)\}\dot{\theta}(t) \\ 0 \end{bmatrix} \tag{31b}$$

where, M_1 and M_2 are the arm mass and the payload mass, respectively, $M_{12}=M_1+M_2$, ℓ is the length of the arm AB. J the net moment of inertia of the arm and the swivel joint is given by

$$J=J_{M1}+J_{M3}=\frac{1}{2}M_3r_z^2+\frac{1}{3}M_1l^2$$

where, J_{M1} and J_{M3} are the moments of inertia of the swivel and the arm respectively, about the z-axis. M_3 and r_z are the mass and the radius of the swivel.

Equation (31) depict a highly coupled nonlinear set of equations. By employing the state dependent transformation

$$u(t)=M(\varphi(t),\gamma)u_t(t) \tag{32}$$

on the input $u(t)$, the equations of motion (30) are transformed into

$$\ddot{\varphi}(t)=-M(\varphi(t),\gamma)f(\varphi,(t),\dot{\varphi}(t),\gamma)+u_t(t) \tag{33}$$

The inertia matrix $M(\varphi(t),\gamma)$ is clearly invertible for all $t\in[0,\infty)$ which follows from the positive definitness of the mass matrix of a manipulator.

Now Eq. (33) can be rewritten in the usual state space form yielding.

$$\begin{aligned} \dot{x}(t)&=\begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix}x(t)+\begin{bmatrix} 0 \\ I_3 \end{bmatrix}u_t(t)+\begin{bmatrix} 0 \\ f_N(x(t),\gamma) \end{bmatrix} \\ y(t)&=[I_3 \quad 0]x(t) \end{aligned} \tag{34}$$

where, I_3 and 0 are 3×3 identity and null matrices respectively,

$$x(t)=\begin{bmatrix} \varphi(t) \\ \dot{\varphi}(t) \end{bmatrix}\in R^6,$$
$$u_t(t)=M^{-1}(\varphi(t),\gamma)u(t)\in R^3,$$

and nonlinear term

$$f_N=-M^{-1}(\varphi(t),\gamma)f(\varphi(t),\dot{\varphi}(t),\gamma)$$

Equation (34) are decoupled with respect to the linear parts and is made use of in executing the design procedure obtained previously in section Ⅱ. This form clearly allows the arbitrary placement of eigenvalues of each decoupled subsystem.

Design Objective :
Our basic design objective is to synthesize a control $u(t)$ in order to achieve the tracking performance specified by the output constraints

$$\|y_i-y_{oi}\|\leq\beta_{oi},\ \ \beta_{oi}>0,\ i=1,2,3$$

despite the payload uncertainty. y_{oi}, $i=1,2,3$ are the 3 nominal outputs to be tracked and y_i are the three actual outputs.

Input Sheres : Let the input spheres be given by

$$\beta_i=1.0,\ \ i=1,2,3$$

Nominal Output :
The nominal outputs to be tracked are

$y_{01}=0.8-0.8e^{-3t}(\cos(t)+3\sin(t))$ for the radial displacement of the arm,
$y_{02}=t^2e^{-t}$ for the angular rotation of the arm,
$y_{03}=0.5-0.5\cos(t)$for the vertical motion of the arm.

Output Spheres :
Output sphere specifications are

$$\beta_{o1}=\beta_{o2}=\beta_{o3}=0.1$$

Thus the tracking specifications call for precise tracking of the nominal outputs given above upto an accuracy of 0.1m in y_1, 0.1rad in y_2 and 0.1m in y_3

Bounded Uncertainty :
We consider the payload M_2 to be the primary uncertainty and assume that

$$M_2\epsilon\Gamma=[0,20]\text{kg}.$$

In order to deign a controller as outlined in the previous section, a threshold value as specified in Eq. (7) needs to be computed first. This requires the computation of several norm quantities as described in the theorem. We use the following data for all computations. $M_1=40$kg, $M_2=[0,20]$ kg, $M_3=100$kg, $\ell=1$m, $r(t)=[0.0,\ 1.0]$m, $z(t)=[0.0,\ 1.0]m$, $\theta(t)=[0,\pi]$rad, and $r_2=0.1$m.
Let the design matrices $V_1=0_3$ and $V_2=I_3$, then
$B_o=[0_3 \quad I_3 \quad 0_3 \quad I_3]^T$

The weighting matrices W_0 and Q_0 chosen primarily to yield favorable norm values are

$$W_0=Q_0=\begin{bmatrix} \Sigma & 0_6 \\ 0_6 & \Sigma \end{bmatrix}$$

where $\Sigma=\begin{bmatrix} I_3 & 0_3 \\ 0_3 & \frac{1}{|\lambda_i|_{\max}}I_3 \end{bmatrix}$ and $|\lambda_i|_{\max}$ is the maximum absolute value of the eigenvalues of matrix R. I_3 is the 3×3 Identity matrix, 0_3 and 0_6 respectively are 3×3 and 6×6 mull matrices.

The selection of the weighting matrices is relatively arbitrary. For example they may be set to the identity matrix. This however will not yield favorable norm values.

Then,

$$\begin{aligned} \rho_0&=\|Q_0B_0\| \\ &=\left\|\begin{bmatrix} \Sigma & 0_6 \\ 0_6 & \Sigma \end{bmatrix}[0_3 \quad I_3 \quad 0_3 \quad I_3]^T\right\| \\ &=\frac{1}{|\lambda_i|_{\max}},\ \ i=1,\cdots,12 \end{aligned}$$

and $\rho_1=\|Q_0B_1\|=0$, since $B_1=0$ due to no external disturbance.

To calculate $\rho_2 = \sup_{\gamma\in\Gamma}\|Q_0(N_\gamma z_0 - N_0 z_0)\|$, we need to select a nominal nonlinear function $g(x)$ to cancel as much as possible the uncertain effects of f_N. we choose $g(x)$ to be of the same form as $f_N(x,\gamma)$ with γ replaced by γ_0, where γ_0 are the arithmetic means of the uncertain parameters. In this case $\gamma = M_2 = [0,20]$kg thus yielding

$$\gamma_0 = \bar{M}_2 = \frac{1}{2}(\underline{M}_2 + \tilde{M}_2) = 10\text{kg}$$

where $\bar{M}_2, \underline{M}_2$ and $\tilde{M}_2$ respectively are the mean value, lower bound and the upper bound of M_2.

Thus, $g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ 0 \end{bmatrix}$

where $g_1(x) = \left(x_1 - \dfrac{M_1 l}{2(M_1 + \bar{M}_2)}\right)x_5^2$

$$g_2(x) = \frac{-2(M_1 + \bar{M}_2)x_1 - M_1 l}{J - M_1 \ell x_1 + (M_1 + \bar{M}_2)x_1^2}x_4 x_5.$$

Thus,

$$\begin{aligned}\rho_2 &= \sup\|Q_0(N_\gamma z_0 - N_0 z_0)\| \\ &= \sup\left\|\begin{bmatrix} I_3 & 0_3 \\ 0_3 & \frac{1}{|\lambda_i|_{\max}} I_3 \end{bmatrix}\begin{bmatrix} 0_3 \\ f_N - f_0(x) \end{bmatrix}\right\| \\ &= \max\left\{\left|\left(\frac{-M_1 l}{2M_{12}} + \frac{M_1 l}{2(M_1 + \bar{M}_2)}\right)\cdot x_5^2\right|\cdot\frac{1}{|\lambda_i|_{\max}}\right. \\ &\left|\left(\frac{-2M_{12}x_1 - M_1 l}{J - M_1 \ell x_1 + M_{12}x_1^2} + \frac{2(M_1 + \bar{M}_2)x_1 + M_2 l}{J - M_1 l x_1 + (M_1 + \bar{M}_2)x_1^2}\right)x_4 x_5\right| \\ &\left.\frac{1}{|\lambda_i|_{\max}}\right\}\end{aligned}$$

On substitution of numerical values, it follows that

$$\rho_2 = \frac{6}{|\lambda_i|_{\max}}$$

Computation of ρ_3 involves the calculation of gradients of the nonlinearity with respect to he state vector x,and is given by

$$\rho_3 = \max\{G_1, G_2\}$$

where,

$$\begin{aligned}G_1 &= \max\|\nabla_x f_{N1}^T\| \\ &= \max\left\{\frac{1}{|\lambda_i|_{\max}}|x_5|, \left|2\left(x_1 - \frac{M_1 l}{2M_{12}}\right)\right|\cdot|x_5|\right\} \\ &= 1.34 \\ G_2 &= \max\|\nabla_x f_{N2}^T\| \\ &= \max\left\{\left|\frac{-2(J - M_1 l x_1 + M_{12}x_1^2)M_{12} + (2M_{12}x_1 + M_1 l)(-M_1 l + 2M_{12})x_1}{(J - M_1 l x_1 + M_{12}x_1^2)^2}\right|\right.: \\ &\frac{1}{|\lambda_i|_{\max}}|x_4 x_5|, \left|\frac{-2M_{12}x_1 - M_1 l}{J - M_1 l x_1 + M_{12}x_1^2}\right|\cdot|x_5|, \\ &\left.\left|\frac{-2M_{12}x_1 - M_1 l}{J - M_1 l x_1 + M_{12}x_1^2}\right|\cdot|x_4|\right\} \\ &= 21.0\end{aligned}$$

Hence, we obtaine $\rho_3 = 21.0$

In computing ρ_2 and ρ_3 as avove it is implicitly assumed that $\frac{1}{|\lambda_i|_{\max}} < 1$. At the end of the design this condition needs to be verified. It will clearly be statisfied in this case.

Now assembling all of the above computations we compute threshold given in Eq. (7)

$$\begin{aligned}&\frac{\beta_0}{\rho_0\beta_i + \rho_1\beta_w + \rho_2 + \rho_3\beta_0} \\ &= \frac{0.1}{1/|\lambda_i|_{\max} + 6.0/|\lambda_i|_{\max} + (21.0)(0.1)}\end{aligned} \tag{35}$$

Now it only remains to fine a set of eigenvalues for the system matrix

$$R = \begin{bmatrix} A & BK \\ -GC & A + BK + GC \end{bmatrix}$$

so that the norm $\|W_0\Psi Q_0^{-1}\|$is less than the upper bound (35). Based on the numerical scheme previously outlined, we obtain the spectra

$$\begin{aligned}\sigma(A+BK) &= \{-47.0\pm j49.0,\ -50.0\pm j53.0, \\ &\quad -53.0\pm j51.0\} \\ \sigma(A+GC) &= \{-110.0\pm j111.0,\ -113.0\pm j114.0, \\ &\quad -115.0\pm j113.0\}\end{aligned}$$

yielding

$$K = \begin{bmatrix} -4160 & 0 & 0 & -94 & 0 & 0 \\ 0 & -5309 & 0 & 0 & -100 & 0 \\ 0 & 0 & -5410 & 0 & 0 & -106 \end{bmatrix}$$

and

$$G = \begin{bmatrix} -220 & 0 & 0 & -24421 & 0 & 0 \\ 0 & -226 & 0 & 0 & -25765 & 0 \\ 0 & 0 & -230 & 0 & 0 & -25994 \end{bmatrix}^T$$

With the abvoe spectra, we obtain the upper bound

$$\frac{\beta_0}{\rho_0\beta_i + \rho_1\beta_w + \rho_2 + \rho_3\beta_0} = 0.045$$

and the critical norm of the operator

$$\|W_0\Psi Q_0^{-1}\| = 0.041$$

which clearly satifies inequality (7).

With K known we can now compute the nominal command input functions $r_{oi}(t), i=1,2,3$, as follows.

$$\begin{aligned}r_{o1}(t) &= 3688 - e^{-3t}(3680\cos(t) + 10336\sin(t)) \\ r_{o2}(t) &= e^{-t}(2 + 196t + 5210t^2) \\ r_{o3}(t) &= 2705 - 2704.5\cos(t) + 53\sin(t).\end{aligned}$$

Thus it follows that the design specified by matrices G,K,the nonlinear function $g(x)$ and the nominal inputs $r_{oi}, i=1,2,3$, guarantee the required tracking performance according to the Theorem 1. The validity of the theorem is also confirmed by simulation results.

Figures (4) ~ (7) show simulations for $M_2 = 20$kg. Figure 4 shows the nominal output y_{01} and the actual output y_1. There is. hardly any difference in the two graphs. This clearly demonstrates the tracking accuracy. Figure 5(a), (b) and

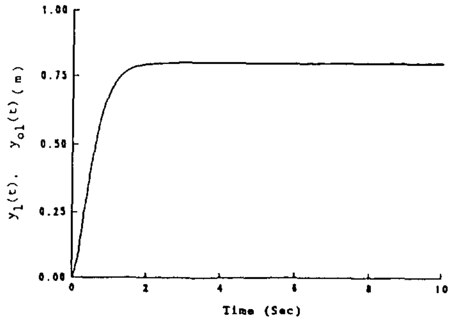

Fig. 4 $y_1(t)$ and $y_{o1}(t)$

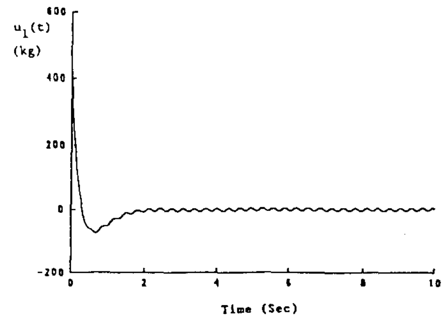

Fig. 6(a) Control effort $u_1(t)$

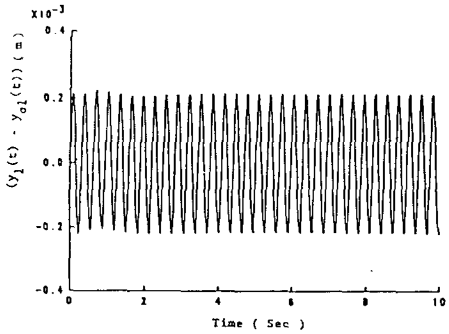

Fig. 5(a) Output deviation $(y_1(t)\text{-}y_0(t))$

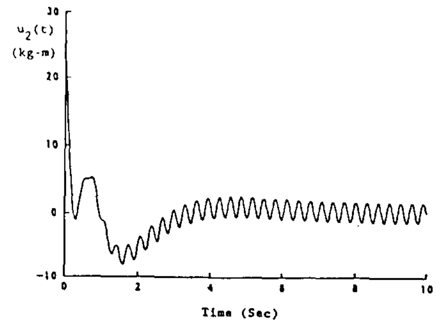

Fig. 6(b) Control effort $u_2(t)$

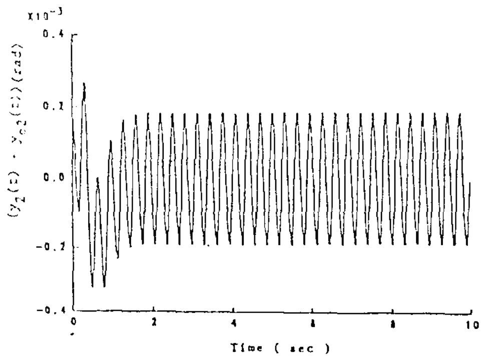

Fig. 5(b) Output deviation$(y_2(t)\text{-}y_{o2}(t))$

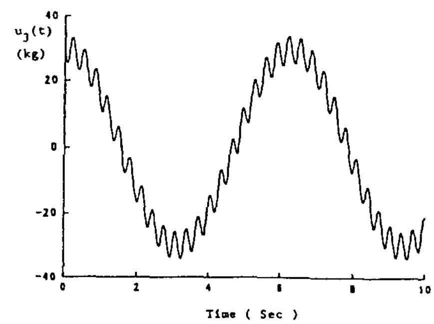

Fig. 6(c) Control effort $u_3(t)$

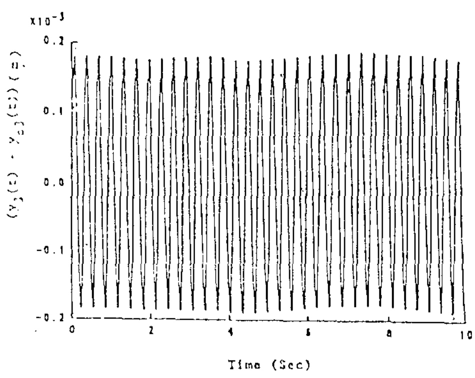

Fig. 5(c) Output deviation $(y_3(t)\text{-}y_{o3}(t))$

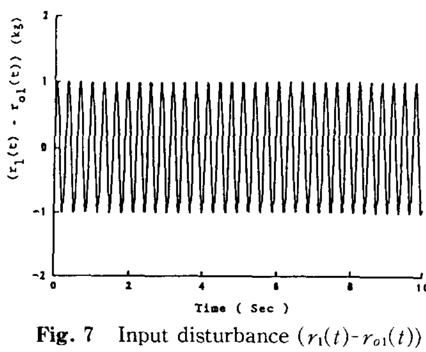

Fig. 7 Input disturbance $(r_1(t)\text{-}r_{o1}(t))$

(c) show the errors $e_i = y_i - y_{oi}, i = 1,2,3$. These errors are of the order of 10^{-3} which is quite conservative in comparison with the imposed output sphere $\beta_0 = 0.1$. This conservativeness is not surprising due to the generality of the inputs and the nonlinearity admissible in $L_\infty[0,\infty)$. The required control inputs are shown in Fig. 6(a), (b), and Fig. 6(c). Figures (8) ~ (10) show simulations for a sinusoidally varying uncertainty $M_2 = 10 + 10\sin(10t)$. Figure 8 shows the nominal output y_{o2} and the actual output y_2. Figure 9 shows the error $y_2 - y_{o2}$ and the control input u_2 is shown in Figure 10. The latter uncertainty is considered just for the sake of demonstrating that the methodology is valid for any uncertainty in

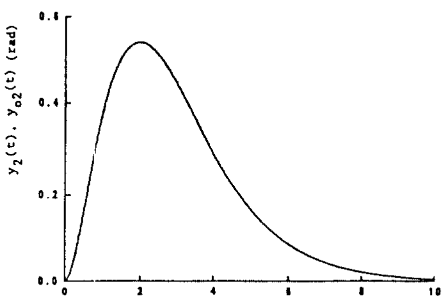

Fig. 8 $y_2(t)$ and $y_{o2}(t)$

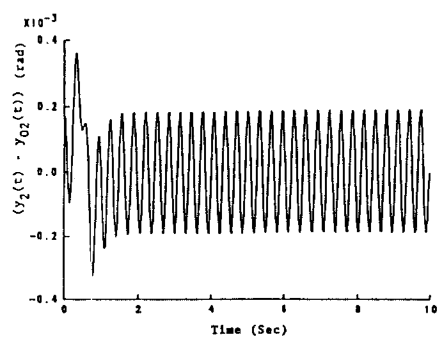

Fig. 9 The tracking error $(y_2(t)\text{-}y_{o2}(t))$

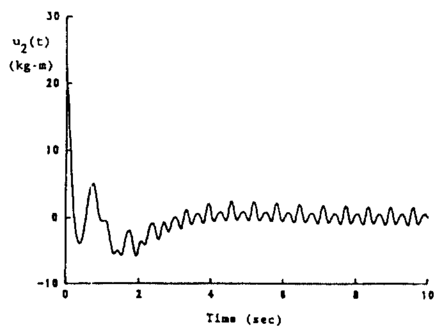

Fig. 10 Control effort $u_2(t)$

a given band.

5. CONCLUSION

In this paper we considered the quantitative pole placement problem central to the direct design of control systems assuring "tracking in the sense of spheres". An approach was proposed to eliminate the adhoc nature of selecting the closed loop poles. It is based on a generalized LQ (linear quadratic) problem formulation. We argue that closed loop pole-patterns that are very close to the Butterworth patterns yield reasonable operator norms for the successful execution of the design criteria stipulated in theorem 1. This is reasonable since in a proper LQ formulation exact Butterworth configurations would yield the optimal L_2-tracking when cheap control (i.e., the weighting on the control tends to zero) is employed.

The basic results were then applied to robotic manipulator for illustrative purposes only. Admittedly there are practical problems which we did not consider in this work. The simulation results clearly demonstrate that the required tracking accuracy is met quite adequately. In fact the design is somewhat conservative. This however is to be expected since any uncertainty whatsoever within the specified bounds is admissible. In particular, any L_∞-function within the given bounds is admissible.

Current research is aimed at obtaining less conservative design criteria by employing L_2-measures for specifying tracking bounds coupled with time varying weighting matrices. Another open problem is to develop a formal theorem to obtain the optimal pole-configuration yielding the minimum closed loop operator norm in an L_∞-setting. The results given in here is a step in this direction.

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